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SOME RESULTS IN DISLOCATED AND DISLOCATED QUASI-METRIC SPACES

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PËRMBLEDHJE
Nocioni i hapësirës së dislokuar metrike është një prej përgjithësimeve të shumta të hapësirës metrike,tek e cila në bazë mbetet parimi i kontraktimit të Banahut . Koncepti i largesës së dislokuar ka zbatime të dobishme në analizën semantike të logjikës së programimit e në shkencat kompjuterike. Në vitin 2005 Zeyada e të tjerë prezantuan konceptin e hapësirës metrike kuazi të dislokuar, si një përgjithësim të hapësirës metrike të dislokuar, gjithashtu përgjithësuan parimin e kontraktimit të Banahut në këto hapësira. Duke përdorur një klasë funksionesh të vazhdueshëm me katër variabla, ne përftojmë disa teorema mbi pikat fikse në hapësirat metrike te dislokuara e kuasi te dislokuara, per nje cift funksjonesh te vazhdueshem dhe per nje funksjon të vetëm.(mbi egzistencën dhe unicitetin). Rezultatet tona zgjerojnë dhe përgjithesojne disa rezultate te koheve te fundit.
Fjalët çelës: largesë e dislokuar, largesë kuazi e dislokuar, pikë fikse, vargu d-konvergjent,varg dq-konvergjent.

SUMMARY
The notion of dislocated metric space is one of the various generalizations of metric space, that retains a variant of the illustrious Banach's Contraction principle and has useful applications in the semantic analysis of logic programmingLater in 2005 Zeyada, F.M. et al introduced the concept of dislocated quasi-metric space and generalized the Banach's Contraction principle in such spaces. The authors there, described the convergence of sequence Cauchy sequence, completeness,.. The purpose of this note is to study and give some fixed point theorems and some generalizations in dislocated metric space. Using a class of continuous functions G_4 we establish fixed point theorems (existence and uniqueness) in dislocated and dislocated quasi-metric spaces, for a pair of continuous mappings and a single mapping. Our results extend and generalize some recently results in the literature.
Key-words: dislocated metric, dislocated quasi-metric, fixed point, d-convergent sequence, dq-convergent sequence

INTRODUCTION

Notion of dislocated metric spaces was introduced by Hitzler and Seda in 2001 as a generalization of metric spaces, where self distances need not to be zero. The concept of dislocated metric space is very useful in logic programming. They generalized famous Banach contraction principle in this space.

In 2005 F.M. Zeyada et al. introduced the concept of dislocated quasi-metric space and generalized the result of Hitzler, P. and Seda, A. K. in such spaces. The authors there described the convergence of sequence in dislocated quasi-metric space and introduced the completeness. Further many authors as Isufati, A [1], and Aage, C. T. & Salunke, J. N. [2], [3], R. Shrivastava, Z. K. Ansari and M. Sharma [6] proved fixed point theorems in dislocated and dislocated quasi-metric spaces, for a single and a pair of continuous mappings.

The purpose of this paper is to prove common fixed point theorems, generalize, give new and improve results in the existing literature, using a class of continuous functions G_4 .

PRELIMINARIES

We start with base and auxiliary definitions and notations, which will be used throughout in this paper.

Definition 2.1 [3] Let X be a nonempty set and let $d: X\times X\to[0,\infty)$ be a function satisfying following conditions:

$d_1; d(x,y)=d(y,x)=0\Rightarrow x=y$

$d_2: d(x,y)\le d(x,z)+d(z,y)$, for all $x,y\in X$.

Then d is called a dislocated quasi-metric on $x_n\to x$. If d satisfies $d(x,x)=0$, for all $x\in X$, then the dislocated quasi-metric is called a quasi-metric on X. If d satisfies $d(x,y)=d(y,x)$ for all $x,y\in X$, then the dislocated quasi-metric is called a dislocated metric on X.

Definition 2.2 [3] A sequence (x_n) in dq-metric space (X,d) is called Cauchy if for all $\varepsilon>0$, $\exists n_0\in N$ such that $\forall m,n\ge n_0$, $d(x_m,x_n)<\varepsilon$ or $d(x_n,x_m)<\varepsilon$.

Definition 2.3[3] A sequence (x_n) dislocated quasi converges or dq-converges to x if $\lim_{n\to\infty} d(x_n,x)=\lim_{n\to\infty} d(x,x_n)=0$. In this case x is called a dq-limit of (x_n) and we write $x_n\to x$.

Definition 2.4 [3] A dq-metric space d, if every Cauchy sequence in it is dq-convergent.

Example 2.5 Let $X=[0,1]$ and $d(x,y)=\max\{x,y\}$. Then the pair (X,d) is a dislocated metric space. We define an arbitrary sequence (x_n) in X by $x_n=\frac{3}{3^n+2}, n\in\mathrm{N}\cup\{0\}$. Let $\varepsilon=\sup_{n\in\mathbb{N}}\left\{\frac{3}{3^n+2}\right\}$, then for $n,m\in\mathrm{N}$ and $n>m$, we have $d(x_n,x_m)=d\left(\frac{3}{3^n+2},\frac{3}{3^m+2}\right)=\frac{3}{3^m+2}\le\varepsilon$. Thus, (x_n) is a Cauchy sequence in X. Also as $n\to\infty$, then $x_n\to 0\in X$. Hence, every Cauchy sequence in X is convergent with respect to d. Thus, (X,d) is a complete dislocated metric space.

Lemma 2.6 [3] Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0.

Definition 2.7 [3] Let (X,d) be a dq-metric space. A mapping $T: X\to X$ is called contraction if there exists $0\le\lambda<1$ such that:

$$d(Tx,Ty)\le\lambda d(x,y \text{ for all } x,y\in X.$$

Lemma 2.8 [4] Let (X,d) be a dq-metric space. If $f: X\to X$ is a contraction function, then $f^n(x_0)$ is a Cauchy sequence for each $x_0\in X$.

Lemma 2.9 [4] dq-limits in a dq-metric space are unique.

MAIN RESULTS

We consider the set G_4 of all continuous functions $g:[0,\infty)^4\to[0,\infty)$ with the following properties:

a) g is non-decreasing in respect to each variable

b) $g(t,t,t,t)\le t, t\in[0,\infty)$

Some examples of these functions are as follows:

$$g_1: g(t_1,t_2,t_3,t_4)=\max\{t_1,t_2,t_3,t_4\}$$

$$g_2: g(t_1,t_2,t_3,t_4)=\max\{t_1+t_2,t_1+t_3,t_2+t_3,t_1+t_4\}$$

$$g_3: g(t_1,t_2,t_3,t_4)=\left[\max\{t_1t_2,t_2t_3,t_3t_1,t_3t_4\}\right]^{\frac{1}{2}}$$

$$g_4: g(t_1,t_2,t_3,t_4)=\left[\max\{t_1^p,t_2^p,t_3^p,t_4^p\}\right]^{\frac{1}{p}}, p>0$$

$$g_5: g(t_1,t_2,t_3,t_4)=c_1t_1+c_2t_2+c_3t_3+c_4t_4$$

$$\text{with } 0\le c_1+c_2+c_3+c_4<1$$

Theorem 3.1 Let (X,d) be a complete dislocated metric space and $T,S: X\to X$ two continuous mappings satisfying the condition:

$$d(Sx,Ty)\le cg\left[\begin{array}{c} d(x,y),d(x,Sx),d(y,Ty), \\ \dfrac{d(x,Sx)d(y,Ty)}{d(x,y)} \end{array}\right] \qquad (1)$$

for all $x,y\in X$ where $g\in G_4$ and $0\le c<1$. Then T and S have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define the sequence (x_n) as follows :

$$x_1=S(x_0), x_2=T(x_1),\dots, x_{2n}=T(x_{2n-1}), x_{2n+1}=S(x_{2n}),..$$

By condition (1) we have:

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d(Sx_{2n},Tx_{2n+1}) \\ &\le cg\begin{bmatrix} d(x_{2n},x_{2n+1}),d(x_{2n},Sx_{2n}), \\ d(x_{2n+1},Tx_{2n+1}),\dfrac{d(x_{2n},Sx_{2n})d(x_{2n+1},Tx_{2n+1})}{d(x_{2n},x_{2n+1})} \end{bmatrix} \\ &= cg\begin{bmatrix} d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}), \\ \dfrac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{d(x_{2n},x_{2n+1})} \end{bmatrix} \\ &\le cd(x_{2n+1},x_{2n}) \end{aligned}$$

Thus

$$d(x_{2n+1},x_{2n+2}) \le cd(x_{2n},x_{2n+1}) \qquad (2)$$

Similarly by condition (1) have :

$$\begin{aligned} d(x_{2n},x_{2n+1}) &= d(Tx_{2n-1},Sx_{2n}) \\ &= d(Sx_{2n},Tx_{2n-1}) \\ &\le cg\begin{bmatrix} d(x_{2n},x_{2n-1}),d(x_{2n},Sx_{2n}), \\ d(x_{2n-1},Tx_{2n-1}),\dfrac{d(x_{2n},Sx_{2n})d(x_{2n-1},Tx_{2n-1})}{d(x_{2n},x_{2n-1})} \end{bmatrix} \\ &= cg\begin{bmatrix} d(x_{2n},x_{2n-1}),d(x_{2n},x_{2n+1}),d(x_{2n-1},x_{2n}), \\ \dfrac{d(x_{2n},x_{2n+1})d(x_{2n-1},x_{2n})}{d(x_{2n},x_{2n-1})} \end{bmatrix} \\ &\le cd(x_{2n-1},x_{2n}) \end{aligned}$$

Thus

$$d(x_{2n},x_{2n+1}) \le cd(x_{2n-1},x_{2n}) \qquad (3)$$

Generally by conditions (2) and (3) have

$$d(x_n,x_{n+1}) \le cd(x_{n-1},x_n) \le \ldots \le c^n d(x_0,x_1) \text{ for } n \in \mathbb{N}$$

Now for $n,m \in N$ with $n<m$, we have:

$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots\ldots + d(x_{m-1},x_m) \\ &\le c^n d(x_0,x_1) + c^{n+1} d(x_0,x_1) + \ldots\ldots + c^{m-1} d(x_0,x_1) \\ &\le \frac{c^n}{1-c} d(x_0,x_1) \end{aligned}$$

Since $0 \le c < 1$, for $n,m \to \infty$ we have $d(x_n,x_m) \to 0$. Hence (x_n) is a Cauchy sequence in complete dislocated metric space (X,d). So there exists $u \in X$ such that (x_n) dislocated converges to $u \in X$. Therefore the subsequences $(Sx_{2n}) \to u$ and $(Tx_{2n+1}) \to u$. Since $T,S : X \to X$ are continuous mappings we get: $Su = u$ and $Tu = u$ Thus, u is a common fixed point of T and S.

Uniqueness : Let suppose that u and v are two fixed of $T;S$ where $Su = u$ and $Tv = v$.

From condition (1) we have:

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\le cg\left[d(u,v),d(u,Su),d(v,Tv),\frac{d(u,Su)d(v,Tv)}{d(u,v)} \right] \\ &= cg\left[d(u,v),d(u,u),d(v,v),\frac{d(u,u)d(v,v)}{d(u,v)} \right] \end{aligned} \qquad (4)$$

Replacing $v=u$ in (4) we get:

$$\begin{aligned} d(u,u) &= d(Su,Tu) \\ &\le cg\begin{bmatrix} d(u,u),d(u,Su),d(u,Tu), \\ \dfrac{d(u,Su)d(u,Tu)}{d(u,u)} \end{bmatrix} \\ &= cg\begin{bmatrix} d(u,u),d(u,u),d(u,u), \\ d(u,u) \end{bmatrix} \\ &\le cd(u,u) \end{aligned}$$

Since $0 \le c < 1$ we obtain $d(u,u) = 0$

Similarly replacing $u=v$ in (4), we obtain $d(v,v) = 0$. Again from (4) have $d(u,v) \le cd(u,v)$ since $0 \le c < 1$ get $d(u,v) = 0$, which implies $u = v$ .Thus fixed point is unique.

The following example illustrates theorem 3.1

Example 3.2 Let $X = [0,1]$ and $d(x,y) = \max\{x,y\}$. Then (X,d) is a dislocated metric space. Define, $Sx = 0, Tx = \frac{x}{6}$. Clearly T,S are continuous and condition (1) is satisfied for all $x,y \in X$, where $\frac{1}{6} \le c < 1$, and take the function $g(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$.

Clearly 0 is the unique common fixed point of T and S.

Corollary3.3 Let (X,d) be a complete dislocated metric space and $T,S : X \to X$ two continuous mappings satisfying the condition:

$$d(Sx,Ty) \le c\max\left\{ \begin{matrix} d(x,y),d(x,Sx),d(y,Ty), \\ \dfrac{d(x,Sx)d(y,Ty)}{d(x,y)} \end{matrix} \right\}$$

for all $x,y \in X$ and $0 \le c < 1$ .Then T and S have a unique common fixed point in X.

This theorem is taken as a corollary of theorem 3.1, if we use the function g_1.

Corollary 3.4 Let (X,d) be a complete dislocated metric space and $T,S:X\to X$ two continuous mappings satisfying the condition:

$$d(Sx,Ty)\le c\max\left\{\begin{array}{l} d(x,y)+d(x,Sx),\\ d(x,y)+d(y,Ty)\\ ,d(x,Sx)+d(y,Ty),\\ d(x,y)+\dfrac{d(x,Sx)d(y,Ty)}{d(x,y)}\end{array}\right\}$$

for all $x,y\in X$ and $0\le c<1$. Then T and S have a unique common fixed point in X.

This theorem is corollary of theorem 3.1 if we use the function g_2.

Corollary 3.5 Let (X,d) be a complete dislocated metric space and $T,S:X\to X$ two continuous mappings satisfying the condition:

$$d^2(Sx,Ty)\le c\max\left\{\begin{array}{l} d(x,y)d(x,Sx),d(x,y)d(y,Ty),\\ d(x,Sx)d(y,Ty),\\ d(y,Ty)\dfrac{d(x,Sx)d(y,Ty)}{d(x,y)}\end{array}\right\}$$

for all $x,y\in X$ and $0\le c<1$. Then T and S have a unique common fixed point in X.

This theorem is corollary of theorem 3.1 if we use the function g_3.

Corollary 3.6 Let (X,d) be a complete dislocated metric space and $T,S:X\to X$ two continuous mappings satisfying the condition:

$$d^p(Sx,Ty)\le c\max\left\{\begin{array}{l} d^p(x,y),d^p(x,Sx),\\ d^p(y,Ty),\left(\dfrac{d(x,Sx)d(y,Ty)}{d(x,y)}\right)^p\end{array}\right\}$$

for all $x,y\in X$ and $0\le c<1$. Then T and S have a unique common fixed point in X.

This theorem is taken as a corollary of theorem 3.1, if we use the function g_4.

Remark 3.7 The theorem3.6 of C.T. Age and J.N. Salunke[2], the Theorem3.7 of R. Shrivastava et al [4] are special case of corollaries 3.3 and 3.4.

Lemma 3.8 Let (X,d) be complete dislocated quasi-metric space and $T:X\to X$ be a continuous mapping, satisfying condition

$$d(Tx,Ty)\le cg\left[\begin{array}{l} d(x,y),d(x,Tx),d(y,Ty),\\ \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)}\end{array}\right] \quad (5).$$

for all $x,y\in X$, where $0\le c<1$ and $g\in G_4$.

If u is a fixed point of T then $d(u,u)=0$

Proof: By condition (5) we have:

$$\begin{aligned} d(u,u)&=d(Tu,Tu)\\ &\le cg\left[\begin{array}{l} d(u,u),d(u,Tu),d(u,Tu),\\ \dfrac{d(u,Tu)d(u,Tu)}{d(u,u)}\end{array}\right]\\ &=cg\left[\begin{array}{l} d(u,u),d(u,u),\\ d(u,u),d(u,u)\end{array}\right]\\ &\le cd(u,u)\end{aligned}$$

Therefore $d(u,u)\le cd(u,u)$ which implies $d(u,u)=0$ since $0\le c<1$.

Theorem 3.9 Let (X,d) be complete dislocated quasi-metric space and $T:X\to X$ a continuous mapping satisfying the condition (5):

$$d(Tx,Ty)\le cg\left[\begin{array}{l} d(x,y),d(x,Tx),d(y,Ty),\\ \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)}\end{array}\right]$$

for all $x,y\in X$, where $g\in G_4$ and $0\le c<1$. Then, T has a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define the sequence (x_n) as follows:

$x_1=T(x_0),x_2=T(x_1),....,x_{n+1}=T(x_n),..$

By condition (5) we have :

$$\begin{aligned} d(x_n,x_{n+1})&=d(Tx_{n-1},Tx_n)\\ &\le cg\left[\begin{array}{l} d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),\\ d(x_n,Tx_n),\dfrac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{d(x_{n-1},x_n)}\end{array}\right]\\ &=cg\left[\begin{array}{l} d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),\\ d(x_n,x_{n+1})\end{array}\right]\\ &\le cd(x_{n-1},x_n)\end{aligned}$$

Thus

$d(x_n,x_{n+1})\le cd(x_{n-1},x_n)$ (6)

Similarly by condition (5) have:

$$d(x_{n-1},x_n)\le cd(x_{n-2},x_{n-1}) \quad (7)$$

Generally by (6) and (7) have

$$d(x_n,x_{n+1}) \le c^n d(x_0,x_1)$$

Taking limit as $n \to \infty$ and since $0 \le c < 1$, we get $d(x_n,x_{n+1}) \to 0$.

Similarly we can show: $d(x_{n+1},x_n) \to 0$.

Hence (x_n) is a dq Cauchy sequence in complete dislocated quasi-metric space (X,d). So there exists $u \in X$ such that (x_n) dislocated quasi converges to u. Since T is a continuous mapping we get:

$T(u) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty}(x_{n+1}) = u$. Thus, u is a fixed point of T.

Uniqueness: Let suppose that $u \ne v$ are two fixed points of T where $Tu = u$ and $Tv = v$.

Using condition (5) and lemma 3.8, we have:

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\le cg\left[d(u,v),d(u,Tu),d(v,Tv),\frac{d(u,Tu)d(v,Tv)}{d(u,v)}\right] \\ &\le cd(u,v) \end{aligned}$$

So $d(u,v) \le cd(u,v)$, since $0 \le c < 1$ get $d(u,v) = 0$.

Similarly we get $d(v,u) = 0$.

Therefore: $d(u,v) = d(v,u) = 0$ implies $u = v$. Hence fixed point is unique.

Corollary 3.10 Let (X,d) be a complete dislocated quasi-metric space and $T: X \to X$ a continuous mapping satisfying the condition:

$$d(Tx,Ty) \le c\max\left\{\begin{array}{l} d(x,y),d(x,Tx),d(y,Ty), \\ \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)} \end{array}\right\}$$

for all $x,y \in X$ and $0 \le c < 1$. Then T has a unique fixed point in X.

Proof. This theorem is taken as corollary of theorem 3.9 if we use the function $g_1 \in G_3$.

Corollary3.11 Let (X,d) be a complete dislocated metric space and $T: X \to X$ two continuous mappings satisfying the condition:

$$d(Tx,Ty) \le c\max\left\{\begin{array}{l} d(x,y)+d(x,Tx), \\ d(x,y)+d(y,Ty) \\ d(x,Tx)+d(y,Ty), \\ d(x,y)+\dfrac{d(x,Tx)d(y,Ty)}{d(x,y)} \end{array}\right\}$$

for all $x,y \in X$ and $0 \le c < 1$. Then T has a unique common fixed point in X.

This theorem is corollary of theorem 3.9 if we use the function g_2.

Corollary 3.12 Let (X,d) be a complete dislocated quasi-metric space and $T: X \to X$ a continuous mapping satisfying the condition:

$$\begin{aligned} d(Tx,Ty) &\le c_1 d(x,y) + c_2 d(x,Tx) \\ &+ c_3 d(y,Ty) + c_4 \frac{d(x,Tx)d(y,Ty)}{d(x,y)} \end{aligned}$$

for all $x,y \in X$ and $0 \le c_1 + c_2 + c_3 + c_4 < 1$. Then T has a unique fixed point in X.

Proof. This theorem is taken as corollary of theorem 3.9 if we use the function $g_5 \in G_4$.

Remark 3.13

- The results of C. T. Aage and J. N. Salunke [3] and A. Isufati [1], theorem 3.3 and corollary 3.1 of R. Shrivastava et al[6], are special case of corollaries 3.10, 3.11, 3.12.
- If in corollary 3.12 we put $c_1 = c_2 = c_3 = c_4$, we obtain a Lipchtic form of contraction.
- Using the functions g_3 and g_4 we can take other corollaries from theorem 3.9.

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