

## REGULARISATION TECHNIQUES FOR FIRST KIND INTEGRAL EQUATIONS

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### PËRMBLEDHJE

Ne kete punim do te paraqesim disa rezultate te reja lidhur me analizen e ekuacioneve integrale te llojit te pare ne teorine e shperhapjes inverse. Me hollesisht, do te trajtojme te ashtuquajturin operator Far field dhe ekuacionin integral korrespondues. Si operatori dhe ekuacioni Far field jane baze per ndertimin e algoritmeve ne teorine e shperhapjes inverse sic eshte metoda e modelimit linear .Gjithashtu do te studiojme ekuacionin integral te llojit te pare qe shfaqet ne metoden e dekompozimit per rikonstruktimin e formes (konturit) ne zgjidhjen e problemit te shperhapjes inverse me pengesa.Do te zbatojme teknika te ndryshme si ajo e Tikhonov dhe e trungezimit per te arritur ne nje zgjidhje te qendrueshme te ketyre ekuacioneve.

### SUMMARY

In this paper we discuss some new results related to the analysis of ill posed first kind integral equations arising in the solution of obstacle inverse scattering theory. More specifically, we investigate the so-called far field operator and the corresponding far field equation. Both the far field operator and the far field equation constitute theoretical basis of uniqueness and reconstruction algorithms in the inverse scattering theory such as the linear sampling method. We study the first kind integral equations associated with the decomposition method and the linear sampling method for solving the inverse obstacle scattering problem. We show how to apply various regularization techniques such as cut-off and Tikhonov regularization to compute a stable solution to these equations.

**Key words:** integral equation, regularization techniques, inverse scattering

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### 1 INTRODUCTION

The field of inverse problems is a relatively new area of mathematical research having its origin in the fundamental papers of Tikhonov in mid-1960s. The reason the area is so young is historical prejudice dating back to Hadamard who claimed that the only problems of physical interest were those that had a unique solution depending continuously on the given data. Such problems were called well-posed, and the problems that were not well – posed were labeled ill-posed. The development of the mathematical theory of ill-posed problems, together with the rapid development of

computing facilities, set the stage for the subsequent mathematical investigation in the inverse problems [3], [6 ]. In this paper we discuss some new results related to the analysis of ill posed first kind integral equations arising in the solution of the obstacle inverse scattering problem. More specifically, we investigate the so called far field operator  $F:L^2[0,2\pi]\rightarrow L^2[0,2\pi]$  and the corresponding far field equation. We also investigate an ill-posed first kind integral equation which appears in the decomposition method for shape reconstruction in the obstacle inverse scattering problem. We show how various regularization techniques such as spectral

cut-off, Tikhonov regularization and the discrepancy principle are applied to regularize the far field equation and how its regularized solution is related to the inverse scattering theory.

**2. ILL-POSED EQUATIONS AND REGULARIZATION TECHNIQUES**

Let  $A : U \rightarrow V$  be an operator, from  $U \in X$  into  $V \in Y$  where  $X, Y$  are normed spaces. The equation  $A\varphi=f$  is called well-posed if  $A$  is bijective and  $A^{-1}:V \rightarrow U$  is continuous. Otherwise  $A\varphi=f$  is called ill-posed.

**Theorem 2.1** Let  $X$  and  $Y$  be normed spaces and let  $A : U \rightarrow V$  be a compact operator. Then  $A\varphi=f$  is ill-posed if  $X$  is not of finite dimension.

**Proof.** Assume  $A^{-1}$  exist and is continuous. Then  $I=A^{-1}A:X \rightarrow X$  is compact hence  $X$  is finite dimension, which ends the proof.

The discontinuity of  $A^{-1}$  leads to the instability of the solution  $\varphi$ . Methods for constructing a stable approximate solution to an ill-posed problem are called regularization methods. In particular, for  $A$  a bounded linear operator, we want to approximate the solution  $\varphi$  of  $A\varphi = f$  from a knowledge of a perturbed right hand side with a known error level  $\|f - f^\delta\| \leq \sigma$ . When  $f \in A(X)$  then if  $A$  is injective there exists a unique solution  $\varphi$  of  $A\varphi=f$ . However, in general we cannot expect that  $f^\delta \in A(X)$ . How do we construct a reasonable approximation  $\varphi^\delta$  to  $\varphi$  that depends continuously on  $f^\delta$ ?

**Definition 2.1** Let  $X$  and  $Y$  be normed spaces and  $A : U \rightarrow V$  be an injective linear bounded operator. Then a family of bounded linear operators  $R_\alpha : Y \rightarrow X, \alpha > 0$  such that  $\lim_{\alpha \rightarrow 0} R_\alpha A\varphi, \forall \varphi \in X$ , is called a regularization scheme for  $A$  and  $\alpha$  the regularization parameter.

We clearly have that  $R_\alpha \rightarrow A^{-1}f$  as  $\alpha \rightarrow 0$ . A regularization scheme approximates the solution  $\varphi$  of  $A\varphi=f$  by  $\varphi_\alpha^\delta := R_\alpha f^\delta$ .

Writing  $\varphi_\alpha^\delta - \varphi = R_\alpha f^\delta - R_\alpha f + R_\alpha A\varphi - \varphi$ , we estimate  $\|\varphi_\alpha^\delta - \varphi\| \leq \delta \|R_\alpha\| + \|R_\alpha A\varphi - \varphi\|$ .

Since the operators  $R_\alpha$  cannot be uniformly bounded with respect to  $\alpha$  and  $R_\alpha A$  cannot be norm convergence as  $\alpha \rightarrow 0$ , the first term on the right hand side is large for  $\alpha$  small whereas the second term on the right hand side is large if  $\alpha$  is not small! So how do we choose  $\alpha$ ? A reasonable strategy is to choose  $\alpha = \alpha(\delta)$  such that  $\varphi_\alpha^\delta \rightarrow \varphi$  as  $\delta \rightarrow 0$ .

**Definition 2.2** A strategy for a regularization scheme  $R_\alpha, \alpha > 0$  is called regular if for every  $f^\delta \in A(X)$  and all  $f^\delta \in Y$  such that

$$\|f - f^\delta\| \leq \sigma \text{ we have that } R_{\alpha(\delta)} f^\delta \rightarrow A^{-1}f \text{ as } \delta \rightarrow 0.$$

A natural strategy for choosing  $\alpha = \alpha(\delta)$  is the discrepancy principle of Morozov, i.e the residual  $\|A\varphi_\alpha^\delta - f^\delta\|$  should not be smaller than the accuracy of the measurements of  $f$ .

From now on  $X$  and  $Y$  will be infinite dimensional and  $A : X \rightarrow Y, A \neq 0$  a compact operator. The operator  $A^* A : X \rightarrow X$  is compact and self-adjoint hence there exists at most a countable set of eigenvalues  $\{\lambda_n\}_1^\infty$ , such that  $A^* A\varphi_n = \lambda_n \varphi_n$ .

Hence  $(A^* A\varphi_n, \varphi_n) = \lambda_n \|\varphi_n\|^2$  which implies that  $\lambda_n \geq 0$ . The non negative square roots of the eigenvalues of  $A^* A$  are called the singular value of  $A$ .

**Theorem 2.2** Let  $\{\mu_n\}_1^\infty$  be the sequence of nonzero singular values of the compact operator  $A : X \rightarrow Y$  ordered such that:  $\mu_1 \geq \mu_2 \geq \mu_3 \dots$

Then there exist orthonormal sequences

$$\{\varphi_n\}_1^\infty \text{ in } X \text{ and } \{g_n\}_1^\infty \text{ in } Y \text{ such that :}$$

$$A\varphi_n = \mu_n g_n, A^* g_n = \mu_n \varphi_n$$

For every  $\varphi \in X$  we have the singular value decomposition:

$$\varphi = \sum_1^{\infty} (\varphi_n, \varphi_n) \varphi_n + P\varphi$$

Where  $P: X \rightarrow N(A)$  is the orthogonal projection operator of  $X$  onto  $N(A)$  and

$$A\varphi = \sum_1^{\infty} \mu_n (\varphi_n, \varphi_n) g_n$$

The system  $(\mu_n \varphi_n g_n)$  is called a singular system of  $A$ .

The following theorem known as Picard's Theorem provides a sufficient condition for the existence of a solution to  $A\varphi = f$  and reveals the ill-posed nature of this equation.

**Theorem 2.3 (Picard's Theorem)** Let  $A: X \rightarrow Y$  be a compact operator with singular system  $(\mu_n \varphi_n g_n)$ . The the equation  $A\varphi = f$  is solvable if and only if  $f \in N(A^*)^{\perp}$  and

$$\sum_1^{\infty} \frac{1}{2\mu_n} |(f, g_n)|^2 < \infty.$$

In this case a solution to  $A\varphi = f$  is given by

$$\varphi = \sum_1^{\infty} \frac{1}{\mu_n} (f, g_n) \varphi_n$$

Note that Picard's Theorem illustrates the ill-posed nature of the equation  $A\varphi = f$ . In particular, settings  $f^{\delta} = f + \delta g_n$ , we obtain a solution of  $A\varphi^{\delta} = f^{\delta}$  given by  $\varphi^{\delta} = \varphi + \delta \varphi_n / \mu_n$ ,

$$\frac{\|\varphi^{\delta} - \varphi\|}{\|f^{\delta} - f\|} = \frac{1}{\mu_n} \rightarrow \infty$$

Since by Hilbert-Schmidt Theorem we have that  $\mu_n \rightarrow 0$ . We say that  $A\varphi = f$  is mildly ill-posed if the singular values decay slowly to zero and severely ill-posed if they decay very rapidly (for example exponentially). All of the inverse scattering problems considered in this book are severely ill-posed.

There are two well-known regular regularization schemes, namely the spectral cut-off method and the Tikhonov regularization.

**Spectral cut-off.** Let  $A: X \rightarrow Y$  be an injective compact operator with singular system  $(\mu_n \varphi_n g_n)$ . Then the spectral cut-off:

$$R_m f := \sum_{\mu_n \geq \mu_m} \frac{1}{\mu_n} (f, g_n) \varphi_n$$

Describes a regularization scheme with regularization parameter  $m \rightarrow \infty$  and  $\|R_m\| = 1/\mu_m$  (see [3] for details)

**Tikhonov Regularization.** Let  $A: X \rightarrow Y$  be a compact operator. Then for every  $\alpha > 0$  the operator  $\alpha I + A^* A: X \rightarrow X$  is bijective and has a bounded inverse. Furthermore, if  $A$  is injective than;

$$R_{\alpha} := (\alpha I + A^* A)^{-1} A^*$$

Describes a regularization scheme, known as Tikhonov regularization with  $\|R_{\alpha}\| \leq \frac{1}{2\sqrt{\alpha}}$ . The

Tikhonov regularization scheme has an equivalent formulation which is formulated in the following theorem (see [3] for the proof).

**Theorem 2.4** Let  $A: X \rightarrow Y$  be a compact operator and let  $\alpha > 0$ . Then for every  $f \in Y$  there exist a unique  $\varphi_{\alpha} \in X$  such that:

$$\|A\varphi_{\alpha} - f\|^2 + \alpha \|\varphi_{\alpha}\|^2 = \inf_{\varphi \in X} \left\{ \|A\varphi - f\|^2 + \alpha \|\varphi\|^2 \right\}$$

The minimize is the unique solution of  $\alpha \varphi_{\alpha} + A^* A \varphi_{\alpha} = A^* f$ .

We finish this section by considering a class of compact integral operators that will appear in the following study of inverse problems. Let

$G \subset \mathbb{R}^m$  be a measurable set.

**Definition 2.3** The linear operator

$$A: L^2(G) \rightarrow L^2(G) \text{ defined by}$$

$$(A\varphi)(x) := \int_G K(x, y) \varphi(y) dy$$

where  $K: G \times G \rightarrow \mathbb{C}$  is a given function known is call an integral operator with kernel  $K$ . If  $K: G \times G \rightarrow \mathbb{C}$  is a continuous function the operator  $A$  is called integral operator with continuous kernel.

The following theorem will be of great importance to us in the following and the proof can be found in [7] and [9].

**Theorem 2.5** The integral operator with continuous kernel is compact in  $L^2(G)$ .

3. Inverse Scattering and Ill-posed Equation

Let us consider the scattering of acoustic plane waves  $u^i := e^{ikx \cdot d}$  in the direction  $d$  by a sound hard obstacle  $D$  (called Scatterer) which for sake of simplicity we assume is a connected bounded region of  $R^2$  at a given fixe frekuensi  $\omega$  where  $k = \omega/c$ ,  $c$  being the sound speed (note that here it assumed that field is time harmonic i.e  $e^{-i\omega t}$  the time dependent term). The scattered

field  $u^s$  satisfies

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } R^2 \setminus \bar{D} \quad (3)$$

$$u^s + e^{ikx \cdot d} = 0 \quad \text{on } \partial D \quad (4)$$

$$\lim_{r \rightarrow \infty} r \left( \frac{\partial u^s}{\partial r} - iku^s \right) = 0 \quad (5)$$

where the Sommerfeld radiation (5) is assumed to hold uniformly in  $\theta$  with  $(r, \theta)$  are polar coordinates. This exterior boundary value problem is well-posed, i.e. a unique solution exists in appropriate spaces [5]. It is known tha the (radiating) fundamental solution to the Helmholtz equation is given by;

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|) \quad (6)$$

where  $H_0^{(1)}$  is the Hankel fuction of the first kind, and note that  $\Phi(x, y)$  satisfies the Sommerfeld radiation condition with respect to both  $x$  and  $y$ .

The scattered field  $u^s$  satisfies the asymptotic the asymptotic behavior [6], [3]

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\theta, \phi) + O(r^{-3/2}) \quad (7)$$

Where  $d = (\cos \phi, \sin \phi)$ ,  $k$  is fixed and

$$u_\infty(\theta, \phi) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} (u^s \frac{\partial}{\partial \nu_y} e^{-ikr_y \cos(\theta - \theta_y)} -$$

$$\frac{\partial u^s}{\partial \nu_y} e^{-ikr_y \cos(\theta - \theta_y)}) ds(y) \quad (8)$$

The function  $u_\infty$  is called the far field pattern corresponding to the scattering problem (3) – (5). Please note that the far field pattern of the fundamental solution

$$\Phi_\infty(\theta, y) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ikr_y \cos(\theta - \theta_y)}, \text{ where}$$

$$y = (r_y, \theta_y)$$

The inverse obstacle scattering problem now is: given the (measured) far field pattern  $u_\infty(\theta, \phi)$ , for  $\theta, \phi \in [0, 2\pi]$  find  $D$ . As we will see bellow this problem is severely ill-posed and non-linear since the far field pattern does not depend linearly on  $D$ . We note that this inverse problem arises from many applications in medical imaging, non-destructive testing, etc. We also note that the exact far field pattern , for  $\theta, \phi \in [0, 2\pi]$  uniquely determine  $D$  [6], [11]. We will present two methods for doing this inverse problems, namely the decomposition method [14] and the linear sampling method [8]. Both methods lead to solving an ill-posed integral equation of the first kind for which we are going to use regularization techniques as developed in Section 2.

3.1 The decomposition method

The main idea of the decomposition method is to break the inverse obstacle scattering problem into two parts: the first part deals with the ill-posedness by constructing the scattered wave  $u^s$  from the far field pattern  $u_\infty$  and the second part deals with the non-linearity by determining the unknown boundary  $\partial D$  of the scatterer as the location where the boundary condition for the total field  $u = e^{ikx \cdot d} + u^s$  is satisfied in a least square sense. We assume that the unknown scatter  $D$  is bounded and simply sonnected and enough a priori information on the unknown scatterer is assumed so that one can place a closed surface  $\Gamma$  inside  $D$ . Then the scattered field  $u^s$  is sought as a single layer potential [9], [13]

$$u^s(x) = \int_{\partial D} \psi(y) \Phi(x, y) ds(y) \quad x \in R^2 \setminus \bar{D} \quad (9)$$

where fundamental solution  $\Phi(\cdot, \cdot)$  is given by (6) and  $\psi \in L^2(\Gamma)$  (the space of square integrable functions in  $\Gamma$ ) is a function to be determined. In

this case the far field pattern  $u_\infty$  has the representation

$$u_\infty(\theta) = \int_{\partial D} e^{-ikr_y \cos(\theta - \theta_y)} \psi(y) ds(y)$$

And  $\psi$  is now determined by solving the integral equation (10). The kernel of the integral operator on the right-hand side of (10) is analytic, whence this is a compact operator according to Theorem 2.5. The later means that (10) is an ill-posed equation, thus in order to solve it one needs to use the Tikhonov (it is known that this operator is injective under some assumption on  $\partial D$ ). Having found the regularized solution  $\psi_\alpha$  with regularization parameter  $\alpha$  and given an approximation of the scattered wave  $u_\alpha^s$  obtained by inserting the Tikhonov regularization solution  $\psi_\alpha$  of (10) into (9), the unknown boundary  $\partial D$  is then determined by requiring that the sound-soft boundary condition  $u^i + u^s = 0$  on  $\partial D$  be satisfied in a least squares sense, i.e by minimizing

$$\|u^i + u_\alpha^s\|_{L^2(\partial D)}^2$$

over a suitable set of admissible curves.

### 3.2 Far field equation and the linear sampling method

We now define the far field operator  $F: L^2[0, 2\pi] \rightarrow L^2[0, 2\pi]$  by

$$(Fg)(\theta) := \int_0^{2\pi} u_\infty(\theta, \phi) g(\phi) d\phi \tag{11}$$

From the representation (8) for  $u_\infty$  and the fact that  $u^s$  depends continuously on  $u^i$  in  $C^1(\partial D)$  we see that  $u_\infty(\theta, \phi)$  is continuous on  $[0, 2\pi] \times [0, 2\pi]$ . This fact combined with Theorem 2.5 proves the following result.

**Theorem 3.1** The far field operator  $F: L^2[0, 2\pi] \rightarrow L^2[0, 2\pi]$  is compact

The far field operator is an important object in the study of inverse obstacle scattering problem considered here. In particular it contains information about the obstacle  $D$  and is related to

the scattering operator  $S$  by  $S = I + \frac{ik}{\sqrt{2\pi k}} e^{\frac{i\pi}{4}} F$ .

By superposition  $Fg$  is the far field pattern of the scattered field due to the Herglotz function

$$v_g := \int_0^{2\pi} e^{ikr \cos(\theta - \phi)} g(\theta) d\phi$$

As incident wave

**Theorem 3.2** The far field operator corresponding to the scattering problem (3) –(5) is injective with dense range, provided that  $k^2$  is not a Dirichlet eigenvalue of  $-\Delta$  in  $D$  (i.e.  $\Delta v + k^2 v = 0$  in  $D$ ,  $v = 0$  on  $\partial D$  has only the trivial solution  $v = 0$ ).

The linear sampling method looks for solution to the far field equation [1], [2], [4], [8], [10].

$$(fg)(\theta) = \Phi_\infty(\theta, z), \quad \text{for } z \in \mathbb{R}^2 \tag{12}$$

To show why the solution of (12) can be used to reconstruct  $D$ , we assume that  $g_z$  solves (12) and  $z \in D$ . Then it follows from Rellich's lemma [3], [6] that

$$\int_0^{2\pi} u^s(x, \phi) g_z(\phi) d\phi = \Phi(x, z) \quad \text{for } z \in \mathbb{R}^2 \setminus D$$

From the boundary condition  $u = 0$  on  $\partial D$  we see that

$$v_{g_z}(x) + \Phi(x, z) = 0 \tag{13}$$

For  $x \in \partial D$  where  $v_{g_z}$  is Herglotz wave function with kernel  $g_z$ . We can now conclude from (13) that  $v_{g_z}$  becomes unbounded as  $z \rightarrow x \in \partial D$  and hence

$$\lim_{z \rightarrow \partial D} \|g_z\|_{L^2[0, 2\pi]} = \infty$$

i.e  $\partial D$  is characterized by points  $z$  where the solution of (12) becomes unbounded. The far field equation is severely ill-posed due to the compactness of the far field operator  $F$ . Thus one solve the regularized equation

$$(\alpha I + F^* F)g = F^* \Phi(\cdot, z), \quad z \in \mathbb{R}^2$$

In fact, only the noisy far field pattern  $u_\infty^\delta(\theta, \phi)$  is known in practice which means that the noisy far field operator  $F_\delta$  is available which is given by

$$(F_\delta g)(\theta) := \int_0^{2\pi} u_\infty^\delta(\theta, \phi) g(\phi) d\phi$$

Where  $\delta$  is the noise level. Thus, one solves the following regularized equation

$$(\alpha(\delta)I + F_{\delta}^* F_{\delta})g = F_{\delta}^* \Phi(\cdot, z), \quad z \in \mathbb{R}^2$$

where the Tikhonov regularization parameter  $\alpha(\delta)$  is chosen by the Morozov discrepancy principle as explained in Section 2.

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