AKTET
Journal of Institute Alb-Shkenca
Reviste Shkencore e Institutit Alb-Shkenca

ISSN 2073-2244
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OSCILLATION OF THE SOLUTIONS TO SECOND ORDER NON-LINEAR DIFFERENTIAL EQUATIONS
OSHILACIONI I ZGJIDHJEVE PËR EKUACIONET DIFERENCIALE JOLINEARE TË RENDIT TË DYTË

ELISABETA KOÇI
Department of Mathematics, Faculty of Natural Sciences, University of Tirana, Tirana, Albania
betakoci@yahoo.com

AKTET V, 3: 315 - 325, 2012

PËRMBLEDHJE
Konsiderojmë ekuacionin diferencial të rendit të dytë jolinear

$$\left(p(t)y'(t)\right)' + q(t)f\left(y(t)\right) = 0 \qquad (1)$$

ku *p* është një funksion me derivat të parë të vazhdueshëm rigorozisht pozitiv, *q* është një funksion i vazhdueshëm për $t \geq t_0$ $t \geq t_0$ dhe *f* është një funksion i vazhdueshëm me vlera reale mbi *R* , i cili kënaq kushtet $yf(y) > 0$ dhe $f'(y) \geq 0$ $f'(y) \geq 0$ për çdo $y \neq 0$ $y \neq 0$.

Qëllimi i këtij punimi konsiston në zgjerimin e rezultateve të Kong [14] dhe Jan Seman [21,22] për rastin jolinear (1) nën supozimin $f'(y) \geq \mu > 0$ për $y \neq 0$ dhe më pas në marrjen e disa kushteve të reja të mjaftueshme për oshilacionin e zgjidhjeve të një rasti të veçante të (1) ku rezultatet e njohura nuk mund të aplikohen. Rezultatet që do të merren do të zbatohen edhe në një shembull, i cili ka vetëm zgjidhje oshilatore për $4km - \beta^2 > 0$. Elementët më të rëndësishëm të përdorur në këtë punim janë teknikat Rikati dhe të mesatarizuara (*averaging*) .
Fjalët çelës: kriter oshilacioni , ekuacion diferencial i rendit të dytë jolinear.

SUMMARY
Consider the second order nonlinear differential equation

$$\left(p(t)y'(t)\right)' + q(t)f\left(y(t)\right) = 0 \qquad (1)$$

Where $p(t) \in C^1([t_0, \infty[, R^+)$ $p(t) \in C([t_0, \infty[, R^+)$, $q(t) \in C([t_0, \infty[, R)$, $t_0 \geq 0$ $q(t) \in C([t_0, \infty[, R)$ and $f \in C(R, R)$ $f \in C(R, R)$ which satisfies the conditions $yf(y) > 0$ and $f'(y) \geq 0$ for all $y \neq 0$.

The purpose of this paper is to extend the results in Kong [14] and Jan Seman [21,22] for the nonlinear case (1) under the assumption $f'(y) \geq \mu > 0$ for all $y \neq 0$ and then getting some new sufficient conditions for oscillatory solutions of a special case of (1) to which the known results do not apply. This result will be implemented also in an example, which has only oscillatory solutions for $4km - \beta^2 > 0$. The important tools used in this paper are the *Riccati* and *averaging* techniques.
Key words: oscillation criteria, second order nonlinear differential equation.

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1. INTRODUCTION

In this paper we will focus mainly on the oscillation behavior of the solutions of second order nonlinear differential equation

$$\left(p(t)y'(t)\right)'+q(t)f\left(y(t)\right)=0 \qquad (1.1)$$

where

$p(t)\in C^1\left([t_0,\infty[,R^+\right)$ $p(t)\in C([t_0,\infty[,R^+)$,

$q(t)\in C\left([t_0,\infty[,R\right), t_0\geq 0$

$q(t)\in C([t_0,\infty[,R)$ and

$f\in C(R,R)$ $f\in C(R,R)$ is continuously differentiable function on the real line R ,except perhaps at point 0 and satisfies the following condition (the condition of its monotonous growth)

$$yf(y)>0 \text{ and } f'(y)\geq 0 \qquad (1.2)$$

where μ is a real constant. In what follows the integral of the function $\frac{1}{p(t)}$ is divergent or convergent (see [21] for comparison).

By a solution of (1.1) we mean a function $y(t)$, $t\in[t_0,\infty)$,which is twice continuously differentiable and satisfies equation (1.1) on $[t_0,\infty)$.Our interest in this paper lies to those solutions of (1.1) that exists on the half - line $[t_0,\infty[$ and satisfy the inequality, $\sup\{|y(t)|:t\geq T\}>0$ for $T\geq t_0$. A nontrivial $y(t)$ solution of (1.1) will be called *oscillatory*, if it has arbitrarily large zeros for all $t\geq t_0$. In other words there exists a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n=\infty$ where $y(t_n)=0$.Otherwise $y(t)$ will be called *nonoscillatory*, therefore there exists a $t_1\geq t_0$ such that $y(t)\neq 0$ for $t\geq t_1$ and in this case the solution $y(t)$ is eventually positive or eventually negative. The equation itself is said to be *oscillatory* if all its solutions are oscillatory.

Especially in these last two decades, oscillation and nonoscillation behavior of the solutions for different classes of the second order differential equations has been the subject of study with different methods by numerous authors. A lot of work has been the object of study for special cases of (1.1) such as these

$$y''(t)+q(t)y(t)=0 \qquad (1.3)$$

$$y''(t)+q(t)|y(t)|^\gamma \operatorname{sgn} y(t)=0 \quad \gamma>0 \qquad (1.4)$$

$$\left(p(t)y'(t)\right)'+q(t)|y(t)|^\gamma \operatorname{sgn} y(t)=0 \quad \gamma>0 \qquad (1.5)$$

Equations (1.4) and (1.5) so-called *Emden-Fowler* equations , considered as prototypes of the respective equations (1.3) and (1.1).

Equations (1.4) and (1.5) are said to be *linear* if $\gamma=1$. Equations (1.1) and [(1.4), (1.5)} are said to be *superlinear* if they satisfy the conditions of superlinearity respectively

$$0<\int_\varepsilon^\infty \frac{dy}{f(y)}<\infty \text{ and } 0<\int_{-\varepsilon}^{-\infty} \frac{dy}{f(y)}<\infty \text{ for all}$$

$$\varepsilon>0 \text{ or } [\gamma>1]$$

Equations (1.1) and [$(1.4),(1.5)$] are said to be *sublinear* if they satisfy the conditions of sublinearity respectively

$$0<\int_0^\varepsilon \frac{dy}{f(y)}<\infty \text{ and } 0<\int_0^{-\varepsilon} \frac{dy}{f(y)}<\infty \text{ for all}$$

$$\varepsilon>0 \text{ or } [0<\gamma<1]$$

In this paper we shall provide conditions which guarantee that all solutions oscillate. We remark that Kong [14], obtained several interval oscillation results for second order differential equation (1.3) or equation (1.1) where $f(y)\equiv y$ .This results however, cannot be applied to the nonlinear equation (1.1). Especially, using phase plane analysis of the Linĕard system , Sugie et.al. [23] discussed the oscillation problem for (1.1) whether the integral of the function $\frac{1}{p(t)}$ is divergent or convergent .

2. SOME OSCILLATION CRITERIA FOR SOLUTIONS OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

In this section, we shall begin with some well-known oscillation criteria for the type (1.3). Most of the oscillation criteria obtained so far involve the interval of the function *q*(*t*) and hence require the information of *q*(*t*) on the entire life-line $\left[t_0,\infty\right[$. In the linear case (1.3) some of the most interesting criteria, in chronological order , are the following:

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(v)\,dv\,ds=\infty$$ (Wintner 1949, [26]) (C 1)

$$\lim_{t\to\infty}\int_{t_0}^{t}q(s)ds=\infty$$ (Leighton 1950, [18]) (C 2)

$$\begin{cases} i) & \liminf\limits_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(v)\,dv\,ds>-\infty \\ ii) & \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(v)\,dv\,ds=\infty \end{cases}$$ Hartman 1952, [11]) (C 3)

$$\lim_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n q(s)ds=\infty$$ for some $n\in N\backslash\{1\}$ (Kamenev 1978,[13]) (C 4)

$$\begin{cases} i) & \limsup\limits_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n q(s)ds<\infty \quad \text{për ndonjë } n\in N\backslash\{1\} \\ ii) & \limsup\limits_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n q(s)ds>B(T) \quad \text{për } \forall T\geq t_0 \\ iii) & \int_{t_0}^{\infty}B_+^2(t)\,dt=\infty \quad \text{ku} \quad B_+(t)=\max\{B(t),0\} \end{cases}$$ (Yan 1986, [24]) (C 5)

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)q(s)-\frac{\left[h(t,s)\right]^2}{4}\right]ds=\infty$$ where $H:D=\left\{(t,s):t\geq s\geq t_0\right\}\to R$

is a continuous function such that $H(t,t)=0$ for $t\geq t_0$, $H(t,s)>0$ for $t>s\geq t_0$ and has a continuous and nonpositive partial derivative on *D* with $\frac{\partial H}{\partial s}(t,s)=-h(t,s)\sqrt{H(t,s)}\leq 0$ for all $(t,s)\in D$, where $h\in C(D,R)$.

(Philos 1989, [19]) (C 6)

For equation (1.4) (Butler [7]) proved that the conditions (C 1) and (C 3) are sufficiently for the superlinear case $(\gamma>1)$ .But for the sublinear case $(0<\gamma<1)$ (Kamenev [12]) proved that only the second part *ii*) of condition (C 3) is enough , without any other condition on $\liminf\limits_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(v)dvds\cdot$

Now the question arises whether we can take the same result for the superlinear case. Willett (1969, [27]) gave a negative response. Again Butler, by comparing the two parts of condition (C3) , obtained another sufficiently condition for the sublinear case.

$$-\infty < \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(v)\,dv\,ds < \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(v)\,dv\,ds \le \infty$$

$$-\infty < \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(v)\,dv\,ds < \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(v)\,dv\,ds \le \infty \qquad (C\,7)$$

Wong (1990 , [28]) also gave another sufficiently condition for the sublinear case

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\left[\int_{t_0}^{s} q(v)\,dv\right]^2 ds = \infty \qquad (C\,8)$$

Li and Yan (1997, [16]) for the superlinear case proved this sufficiently condition

$$\begin{cases} \liminf_{t\to\infty} \frac{1}{t^{\beta}}\int_{t_0}^{t}(t-s)^{\beta} q(s)ds > -\infty & \text{për ndonjë } \beta \ge 1 \\ \limsup_{t\to\infty} \frac{1}{t^{\alpha}}\int_{t_0}^{t}(t-s)^{\alpha} q(s)ds = \infty & \text{për ndonjë } \alpha > 1 \end{cases} \quad \text{ku } \alpha,\beta \in N$$

But many of these conditions cannot be applied to the case when $q(t)$ has a " bad " behavior on a large part of $[t_0,\infty[$, for example, when $\lim_{t\to\infty}\int_{t_0}^{t} q(s)ds = -\infty$

3. MAIN RESULTS

In this section, we present our main theorems.

Theorem 3.1 . Assume that $f'(y) \ge \mu > 0$ for all $y \ne 0$. .

Let $D_0 = \{(t,s) : t > s \ge t_0\}$, and $H \in C(D,R)$ $H \in C(D,R)$ which satisfies the following conditions

i) $H(t,t) = 0$, for $t \ge t_0$ and $H(t,s) > 0$ for $t > s \ge t_0$ (on D_0),

ii) Function *H* has continuous and partial derivative on D_0 with respect both variables *t* and s.

iii) $\frac{\partial H}{\partial t}(t,s) = h_1(t,s)\sqrt{H(t,s)} \ge 0$ and $\frac{\partial H}{\partial s}(t,s) = -h_2(t,s)\sqrt{H(t,s)} \le 0$

where h_1 h_1 and h_2 are continuous and nonnegative on D_0.

If there exists a function such that

$$\frac{1}{H(c,a)}\int_a^c H(s,a)g(s)q(s)ds + \frac{1}{H(b,c)}\int_c^b H(b,s)g(s)q(s)ds > \qquad (3.1)$$

$$> \frac{1}{4\mu H(c,a)}\int_a^c \left[Q_1(s,a)\right]^2 g(s)p(s)ds + \frac{1}{4\mu H(b,c)}\int_c^b \left[Q_2(b,s)\right]^2 g(s)p(s)ds$$

where

$$Q_1(s,t) = h_1(s,t) + \frac{g'(s)}{g(s)}\sqrt{H(s,t)}$$

$$Q_2(t,s) = h_2(t,s) - \frac{g'(s)}{g(s)}\sqrt{H(t,s)}$$

then all solutions of (1.1) are oscillatory.

Proof. Without any loss of generality , we may assume that there exists a solution $y(t)$ for our equation such that on $[T_0,\infty]$ for some sufficiently large . We build the Riccati support function $u(t)$ as follows :

$$u(t)=g(t)p(t)\left\{\frac{y'(t)}{f(y(t))}\right\} \text{ for all } t\geq T_0$$

Differentiating the above equation we obtain

$$=-q(t)f(y(t))\left\{\frac{g(t)}{f(y(t))}\right\}+y'(t)p(t)\left[\frac{g(t)}{f(y(t))}\right]=-q(t)g(t)+y'(t)p(t)\left[\frac{g'(t)f(y(t))-g(t)f'(y)y'(t)}{f^2(y(t))}\right]=$$

$$=-q(t)g(t)+\frac{g'(t)}{g(t)}u(t)-\frac{f'(y)}{p(t)g(t)}u^2(t)\leq -q(t)g(t)+\frac{g'(t)}{g(t)}u(t)-\frac{\mu}{p(t)g(t)}u^2(t)$$

Thus we get the following inequality

$$q(t)g(t)\leq -u'(t)+\frac{g'(t)}{g(t)}u(t)-\frac{\mu}{p(t)g(t)}u^2(t) \qquad (3.2)$$

Multiplying to the above inequality by the function $H(t,s)$ and integrating it with respect to s , over $[c,t)$ for $t\in[c,b)$ we get for $s\in[c,t)$

Integrating by parts the first integral of *A*, we have

$$A=H(t,c)u(c)-\int_c^t u(s)h_2(t,s)\sqrt{H(t,s)}ds+\int_c^t H(t,s)\left[\frac{g'(s)}{g(s)}u(s)-\frac{\mu}{p(s)g(s)}u^2(s)\right]ds=$$

$$=H(t,c)u(c)-\int_c^t \left\{\left[\frac{\mu H(t,s)}{p(s)g(s)}\right]^{\frac{1}{2}}u(s)-\frac{1}{2}\left[\frac{p(s)g(s)}{\mu}\right]^{\frac{1}{2}}Q_2(t,s)\right\}^2 ds+\int_c^t \frac{p(s)g(s)}{4\mu}Q^2{}_2(t,s)ds\leq$$

$$\leq H(t,c)u(c)+\int_c^t \frac{p(s)g(s)}{4\mu}Q^2{}_2(t,s)ds$$

From the above inequality, we have

$$\int_c^t H(t,s)g(s)q(s)ds\leq H(t,c)u(c)+\int_c^t \frac{p(s)g(s)}{4\mu}Q^2{}_2(t,s)ds$$

Passing to the limit for $t\to b^-$ on both sides and also dividing by

$$\frac{\lim_{t\to b^-}\int_c^t H(t,s)g(s)q(s)ds}{H(b,c)} \le \frac{\lim_{t\to b^-} H(t,c)u(c)+\lim_{t\to b^-}\int_c^t \frac{p(s)g(s)}{4\mu}Q^2{}_2(t,s)ds}{H(b,c)}$$

This implies that

$$\frac{\int_c^b H(b,s)g(s)q(s)ds}{H(b,c)} \le \frac{H(b,c)u(c)+\int_c^b \frac{p(s)g(s)}{4\mu}Q^2{}_2(b,s)ds}{H(b,c)} \qquad (3.3)$$

On the other hand , by multiplying the function $H(s,t)$ in inequality (3.2) and integrating it with respect to s, over $[t,c]$ for $t\in[a,c]$ us to take for

So define

$$B=-H(c,t)u(c)+\int_t^c u(s)h_1(s,t)\sqrt{H(s,t)}ds+\int_t^c H(s,t)\left[\frac{g'(s)}{g(s)}u(s)-\frac{\mu}{p(s)g(s)}u^2(s)\right]ds=$$

$$=-H(c,t)u(c)-\int_t^c \left\{\left[\frac{\mu H(s,t)}{p(s)g(s)}\right]^{\frac{1}{2}}u(s)-\frac{1}{2}\left[\frac{p(s)g(s)}{\mu}\right]^{\frac{1}{2}}Q_1(s,t)\right\}^2 ds+\int_t^c \frac{p(s)g(s)}{4\mu}Q^2{}_1(s,t)ds\le$$

$$\le H(c,t)u(c)+\int_t^c \frac{p(s)g(s)}{4\mu}Q^2{}_1(s,t)ds$$

Finally we have

$$\int_t^c H(s,t)g(s)q(s)ds\le -H(c,t)u(c)+\int_t^c \frac{p(s)g(s)}{4\mu}Q^2{}_1(s,t)ds$$

Passing to the limit on both its sides for $t\to a^+$ and also dividing by $H(c,a)\ne 0$

$$\frac{\lim_{t\to a^+}\int_t^c H(s,t)g(s)q(s)ds}{H(c,a)} \le \frac{\lim_{t\to a^+} -H(c,t)u(c)+\lim_{t\to a^+}\int_t^c \frac{p(s)g(s)}{4\mu}Q^2{}_1(s,t)ds}{H(c,a)}$$

$$\frac{\int_a^c H(s,a)g(s)q(s)ds}{H(c,a)} \le \frac{-H(c,a)u(c)+\int_a^c \frac{p(s)g(s)}{4\mu}Q^2{}_1(s,a)ds}{H(c,a)} \qquad (3.4)$$

Combining the two inequalities (3.3) and (3.4), we obtain

$$\frac{\int_a^c H(s,a)g(s)q(s)ds}{H(c,a)}+\frac{\int_c^b H(b,s)g(s)q(s)ds}{H(b,c)} \le \frac{\int_a^c \frac{p(s)g(s)}{4\mu}Q^2{}_1(s,a)ds}{H(c,a)}+\frac{\int_c^b \frac{p(s)g(s)}{4\mu}Q^2{}_2(b,s)ds}{H(b,c)}$$

which contradicts (3.1). The proof is complete.

Remark 3.2

In many other articles it is assumed that $q(t) > 0$ for $t \geq t_0$. Our Theorem 3.1, does not exclude the case when $q(t) < 0$ for $t \geq t_0$.

Remark 3.3

One of the most popular appearances of the function $H(t,s)$ is $H(t,s) = (t-s)^{\lambda}$ where $t \geq s \geq t_0$ and $\lambda > 1$ is a constant.

For the other choices of $H(t,s)$ condition (3.1) reduces to the other type conditions (see [12]).

Theorem 3.4 Assume that the conditions of Theorem 3.1 are satisfied. If there exists a function $g(t) \in C^1([t_0, \infty[, R^+)$ and for each $l \geq t_0$ the following conditions

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \left[\int_l^t (s-l)^{\lambda} g(s) q(s)\, ds - \int_l^t \frac{(s-l)^{\lambda-2}}{4\mu} g(s) p(s) \left[\lambda + \frac{g'(s)(s-l)^2}{g(s)} \right] ds \right] > 0, \quad (3.5)$$

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \left[\int_l^t (t-s)^{\lambda} g(s) q(s)\, ds - \int_l^t \frac{(t-s)^{\lambda-2}}{4\mu} g(s) p(s) \left[\lambda + \frac{g'(s)(t-s)^2}{g(s)} \right] ds \right] > 0, \quad (3.6)$$

are satisfied, then all solutions of (1.1) are oscillatory.

Proof . We get after substitution

$$[Q_1(s,t)]^2 = \left[\frac{\frac{\partial H}{\partial s}(s,t)}{\sqrt{H(s,t)}} + \frac{g'(s)}{g(s)} \sqrt{H(s,t)} \right]^2 = \left[\lambda (s-t)^{\lambda-1-\frac{\lambda}{2}} + \frac{g'(s)}{g(s)} (s-t)^{\frac{\lambda}{2}} \right]^2 =$$

$$= \left[(s-t)^{\frac{\lambda-2}{2}} \left\{ -\lambda + \frac{g'(s)}{g(s)} (s-t) \right\} \right]^2 = (s-t)^{\lambda-2} \left\{ \lambda + \frac{g'(s)}{g(s)} (s-t) \right\}^2$$

$$[Q_2(t,s)]^2 = \left[\frac{-\frac{\partial H}{\partial s}(t,s)}{\sqrt{H(t,s)}} - \frac{g'(s)}{g(s)} \sqrt{H(t,s)} \right]^2 = \left[\lambda (t-s)^{\lambda-1-\frac{\lambda}{2}} - \frac{g'(s)}{g(s)} (t-s)^{\frac{\lambda}{2}} \right]^2 =$$

$$= \left[(t-s)^{\frac{\lambda-2}{2}} \left\{ \lambda - \frac{g'(s)}{g(s)} (t-s) \right\} \right]^2 = (t-s)^{\lambda-2} \left\{ \lambda - \frac{g'(s)}{g(s)} (t-s) \right\}^2$$

Let us replace in the inequality (3.5) $l = a$ where $a \geq t_0$.

Then there exists $c > a$ such that

$$\limsup_{c\to\infty} \frac{1}{c^{\lambda-1}} \left[\int_a^c (s-a)^{\lambda} g(s) q(s)\, ds - \int_a^c \frac{(s-a)^{\lambda-2}}{4\mu} g(s) p(s) \left[\lambda + \frac{g'(s)(s-a)^2}{g(s)} \right] ds \right] > 0$$

Let us replace in the inequality (3.6) $l = c$ where $c \geq t_0$.

Then exists $b > c$ where have

$$\limsup_{b\to\infty} \frac{1}{b^{\lambda-1}} \left[\int_c^b (b-s)^{\lambda} g(s) q(s)\, ds - \int_c^b \frac{(b-s)^{\lambda-2}}{4\mu} g(s) p(s) \left[\lambda + \frac{g'(s)(b-s)^2}{g(s)} \right] ds \right] > 0$$

By combining the above two inequalities we get condition (3.1). So, the conclusion now follows from Theorem 3.1.

Theorem 3.5 Assume that the conditions of Theorem 3.1 be satisfied, by adding the condition $q(t)>0$ for all $T\ge t_0$. Assume further there exists a function and $w\in C[a,b]$ that satisfies the conditions $w'(t)\in L^2[a,b]$ and $w(a)=w(b)=0$. If we have

$$\int_a^b\left\{w^2(s)q(s)g(s)-\frac{g(s)p(s)}{\mu}\left[w'(s)+\frac{1}{2}\frac{g'(s)}{g(s)}w(s)\right]^2\right\}ds>0 \qquad (3.7)$$

then equation (1.1) is oscillatory.

Proof. Without loss of generality , we may assume that there exists a solution $y(t)$ for (1.1), such that on $[T_0,\infty]$ for some sufficiently large . As in Theorem 3.1 in inequality (3.2) we multiply and have

$$q(t)g(t)w^2(t)\le -u'(t)w^2(t)+\frac{g'(t)}{g(t)}u(t)w^2(t)-\frac{\mu}{p(t)g(t)}u^2(t)w^2(t) \qquad (3.8)$$

Integrate with respect to *s,* using the condition . Then we get

$$\int_a^b q(s)g(s)w^2(s)ds\le\int_a^b -u'(s)w^2(s)ds+\int_a^b w^2(s)\left\{\frac{g'(s)}{g(s)}u(s)-\frac{\mu}{p(s)g(s)}u^2(s)\right\}ds=$$

$$=2\int_a^b u(s)w(s)w'(s)ds+\int_a^b w^2(s)\left\{\frac{g'(s)}{g(s)}u(s)-\frac{\mu}{p(s)g(s)}u^2(s)\right\}ds \qquad (3.9)$$

Use the following substitutions

$$a_1(s)=\frac{p(s)g(s)}{\mu} \quad \text{and} \quad a_2(s)=\frac{g'(s)}{g(s)}$$

$$A(s)=\sqrt{\frac{p(s)g(s)}{\mu}}\left[w'(s)+\frac{1}{2}\frac{g'(s)}{g(s)}w(s)\right]=\sqrt{a_1(s)}\left[w'(s)+\frac{1}{2}a_2(s)w(s)\right]$$

$$B(s)=\sqrt{\frac{\mu}{p(s)g(s)}}u(s)w(s)-\sqrt{\frac{p(s)g(s)}{\mu}}\left[w'(s)+\frac{1}{2}\frac{g'(s)}{g(s)}w(s)\right]$$

$$\frac{u(s)w(s)}{\sqrt{a_1(s)}}-\sqrt{a_1(s)}\left[w'(s)+\frac{1}{2}a_2(s)w(s)\right]=\frac{u(s)w(s)}{\sqrt{a_1(s)}}-A(s)$$

$$A^2(s)-B^2(s)=\left[2A(s)-\frac{u(s)w(s)}{\sqrt{a_1(s)}}\right]\left[\frac{u(s)w(s)}{\sqrt{a_1(s)}}\right]=2A(s)\frac{u(s)w(s)}{\sqrt{a_1(s)}}-\left[\frac{u(s)w(s)}{\sqrt{a_1(s)}}\right]^2$$

$$2u(s)w(s)\left[w'(s)+\frac{1}{2}a_2(s)w(s)\right]-\left[\frac{u(s)w(s)}{\sqrt{a_1(s)}}\right]^2=$$

$$=2u(s)w(s)w'(s)+u(s)w^2(s)\frac{g'(s)}{g(s)}-\frac{\mu}{p(s)g(s)}u^2(s)w^2(s)$$

From inequality (3.9) we have

$$\int_a^b q(s)g(s)w^2(s)ds\le\int_a^b\left[A^2(s)-B^2(s)\right]\le\int_a^b A^2(s)ds=$$

$$=\int_a^b\left\{\sqrt{\frac{p(s)g(s)}{\mu}}\left[w'(s)+\frac{1}{2}\frac{g'(s)}{g(s)}w(s)\right]\right\}^2ds=\int_a^b\frac{p(s)g(s)}{\mu}\left[w'(s)+\frac{1}{2}\frac{g'(s)}{g(s)}w(s)\right]^2ds$$

which contradicts assumption (3.7), so equation (1.1) is oscillatory.

The following example illustrates our results.

Example 3.6. Consider the differential equation i^* (*special case: free ,damped motion with the damping coefficient* $\beta > 0$, $m > 0$, $k > 0$ *for all* $t \geq 0$) . Differential equation i^* reduses to ii^* and iii^* as follows :

$i^*)\quad x'' + \frac{\beta}{m}x' + \frac{k}{m}x = 0$,

$ii^*)\quad u'' + \left(\frac{k}{m} - \frac{1}{4}\frac{\beta^2}{m^2}\right)e^{\frac{\beta}{m}t}u = 0$

$iii^*)\quad \left(p(t)x'(t)\right)' + q(t)x(t) = 0$

Here, $p(t) = e^{\frac{\beta}{m}t}$, $q(t) = \frac{k}{m}e^{\frac{\beta}{m}t}$, $f(x) = x,\ u(t) = e^{\frac{1}{2}\frac{\beta}{m}t}x(t)$ and $f_x'(x) = \mu = 1$

- $\int_0^\infty \frac{ds}{p(s)} = \int_0^\infty e^{-\frac{\beta}{m}s}ds = \frac{m}{\beta} < \infty,$
- By (C 2) , equation ii^* is oscillatory for $\frac{k}{m} - \frac{1}{4}\frac{\beta^2}{m^2} > 0\ \left[4km - \beta^2 > 0\right]$ and $\frac{m}{\beta} > 0$

$$\lim_{t\to\infty}\int_0^t q(s)\,ds = \lim_{t\to\infty}\int_0^t \left(\frac{k}{m} - \frac{1}{4}\frac{\beta^2}{m^2}\right)e^{\frac{\beta}{m}s}ds = \lim_{t\to\infty}\left(\frac{k}{m} - \frac{1}{4}\frac{\beta^2}{m^2}\right)\frac{m}{\beta}\left(e^{\frac{\beta}{m}t} - 1\right) = +\infty$$

- While by Theorem 3.1, we have for iii^*

$$H(t,s) = e^{\frac{\beta}{m}(t-s)},\quad Q_1(t,s) = h_1(t,s) = Q_2(t,s) = h_2(t,s) = \frac{\beta}{m}e^{\frac{1}{2}\frac{\beta}{m}(t-s)},\quad g(s) = 1$$

$$\int_a^c H(s,a)\frac{k}{m}e^{\frac{\beta}{m}s}ds + \int_c^b H(b,s)\frac{k}{m}e^{\frac{\beta}{m}s}ds > \frac{1}{4}\int_a^c H(s,a)\frac{\beta^2}{m^2}e^{\frac{\beta}{m}s}ds + \frac{1}{4}\int_c^b H(b,s)\frac{\beta^2}{m^2}e^{\frac{\beta}{m}s}ds\ ,$$

$$\frac{k}{m}\left[\int_a^c e^{\frac{\beta}{m}(s-a)}e^{\frac{\beta}{m}s}ds + \int_c^b e^{\frac{\beta}{m}(b-s)}e^{\frac{\beta}{m}s}ds\right] > \frac{1}{4}\frac{\beta^2}{m^2}\left[\int_a^c e^{\frac{\beta}{m}(s-a)}e^{\frac{\beta}{m}s}ds + \int_c^b e^{\frac{\beta}{m}(b-s)}e^{\frac{\beta}{m}s}ds\right]$$

$$\left[\frac{k}{m} - \frac{1}{4}\frac{\beta^2}{m^2}\right]\left[\int_a^c e^{\frac{\beta}{m}(2s-a)}ds + \int_c^b e^{\frac{\beta}{m}(b-s)}e^{\frac{\beta}{m}s}ds\right] > 0$$

So for $4km - \beta^2 > 0$ and $\frac{m}{\beta} > 0$, equation $iii^*)$ is oscillatory by the Theorem 3.1 .

- Theorem 3.5, when applied to iii^* , leads to the following results.

$w(s) = \sin s\quad (\sin 2k\pi = \sin 2k\pi + \pi = 0)$, $g(s) = 1$ or $\left[g(s) = e^{-\frac{\beta}{m}s}\right]$ then equation $iii^*)$ is oscillatory where $2km - 2m^2 > \beta^2$.

REFERENCES

[1] Agarwal, R. P., Grace, S. R., O'Regan ,D., (2000), *Oscillation Theory for Difference and Functional Differential Equations* , Kluwer Academic Publishers.

[2] Agarwal, R. P., Grace, S. R. , O'Regan ,D., (2002), *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations* , Kluwer Academic Publishers.

[3] Agarwal, R. P., Bohner, M., Li, W. T.,(2004), *Nonoscillation and Oscillation Theory for Functional Differential Equations* , Marcel Dekker.

[4] Bahatia, N. B., (1966), *Some Oscillation Theorems for Second Order Nonlinear Equations*, J. Math. Anal. Appl., Vol.15, 442-446.

[5] Bin, Zh.*, Interval Oscillation Criteria Related to interval Averaging Technique* ***f****or Certain Nonlinear Delay Differential Equations,* (2008), Int. Journal of Math Analysis, Vol.2, No .4 , 149 – 157.

[6] Bohner, M. , Saker, S. H., *Oscillation of Damped Second Order Nonlinear Delay Differential Equations of Emden-Fowler Type* ,(2006) ,Advances in Dynamic Sys. Appl, Vol.1, No. 2 , 163-182.

[7] Butler,G,J., *Integral averages and the oscillation of second order ordinary differential equations ,* (1980) , SIAM J. Math.Anal. No.11, 190-200.

[8] Elabbasy, E. M., Elzeiny,Sh. R. , *Oscillation Theorems concerning Non-Linear Differential Equations of the Second Order* , (2011) ,Opuscula Mathematica, Vol.31, No. 3 , 373-389.

[9] El-Sayed,M.A., *An oscillation criterion for a forced second order linear differential equations*, (1993) Proc. Am. Math. Soc, Vol.118, 813-817.

[10] Erbe, L., Hasan,T. S., Peterson, A. , *Oscillation of Second Order Neutral Delay Differential Equations* , (2008), Advances in Dynamic Sys. Appl, Vol. 3, No. 1, 53-71.

[11] Hartman, P., *Non oscillatory linear differential equations of second order*,(1952), Amer.J.Math, No.74, 389-400.

[12] Kamenev, I, V., *Certain specifically nonlinear oscillation theorems.*(1971), Mat.Zametki, No.10, 129-134. *In Russian*.

[13] Kamenev, I, V., *An integral criterion for oscillation of linear differential equations of second order.*(1978), Mat.Zametki, No.23, 249-1251. *In Russian*.

[14] Kong,Q., *Interval criteria for oscillation of second order linear ordinary differential equations*, (1999), J. Math. Anal. Appl,Vol.229, 258-270.

[15] Li ,W. T., *Interval oscillation of second-order half-linear function differential equations* , (2004), Applied Mathematics and Computation, Vol.155 **,** 451-468.

[16] Li ,W. T.,Yan, J., *Oscillation Criteria for Second Order Superlinear Differential Equations* ,(1997), Indian J.Pure. Appl.Math, Vol.28 , 735-740.

[17] Li, W. T. , Huo, H. F. , *Interval Oscillation Criteria for Nonlinear Second Order Differential Equations* ,(2001), Indian J.Pure. Appl.Math, Vol.32 , No. **7,** 1003-1014.

[18] Leighton, W., *The dedection of the oscillations of solutions of a second order linear differential equation ,* (1950) , Duke Math.J., No.17, 57-62.

[19] Philos,Ch,G., *Oscillation theorems for linear differential equations of second order,* (1989), Arch.Math. No.53, 482-492.

[20] Saker,S. H., Pang,P. Y. H., Agarwal,R. P., *Oscillation Theorems for Second Order Nonlinear Functional Differential Equations with Damping ,* (2003), Dynamic Sys. Appl, Vol. 12 **,** 307-322.

[21] Seman ,J., *Oscillation Theorems for Second Order Nonlinear Delay Inequalities* (1989), Math. Slovaca, Vol. 39 , No. 3, 313-322.

[22] Seman ,J., *Oscillation of solutions to Second Order linear Differential Equations* , (2004), Electronic Journal of Differential Equations, Vol. 2004, No.28 , 1-9.

[23] Sugie, J., Kita, K ., Yamaoka, N., *Oscillation constant of second-order non-linear self- adjoint differential equations* , (2002), Ann.Mat. Pura Appl ,Vol.4, No.181, 309-337

[24] Wan, J., *Oscillation Theorems for Second Order linear Differential Equations with Damping Term,* (1986) , Proc.Amer.Math.Soc. No.98, 276-282.

[25] Wang, X ., Song ,G., *Oscillation Criteria for a Second -Order Nonlinear Damped Differential Equation* , (2011), Intern J of ISC ,Vol.7, No. 1 , 73-82.

[26] Wintner, A. *A criterion of oscillatory stability.*(1949), Quart.Appl.Math., No.7, 115-117.

[27] Willet, D,W., *On the oscillatory behavior of the solutions of second order linear differential equations,*(1969) , Ann.Polon.Math, No.21, 175-191.

[28] Wong J.S.W., *An oscillation theorem for second order sublinear differential equations* ,(1990), Proc.Amer.Math.Soc, No.110, 633-637.

[29] Yamaoka N.,*Oscillation and Nonoscillation Theorems for Second Order Nonlinear Differential*

Equations with p- Laplacian,(2005),Mathematical Science Vol.38, 17-30 .
[30] Çakmak, D., *Oscillation for Second Order Nonlinear Differential Equations with Damping,* (2008), Dynamic Sys. Appl,Vol.17, 139-148.

1. [31] Zaki, M. S., *On asymptotic behavior of a second order delay differential equation* (2007), Intern. J of Physical Sciences ,Vol.2 , No. 7, 185-187.